DIGEST OF THE CARTAN PAPER BY OZAWA AND POPA (New development of Operator Algebras)

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DIGEST OF THE CARTAN PAPER BY OZAWA AND POPA

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ABSTRACT. This is a digest of the paper [OP] by S. Popa and the author. We prove that the normalizer of any diffuse amenable subalgebra of a free group factor $L(\mathbb{F}_r)$ generates an amenable von Neumann subalgebra. We also sketch the proof of the fact that if a free ergodic measure preserving action of a free group $\mathbb{F}_r$, $2 \leq r \leq \infty$, on a probability space (X, μ) is profinite then the group measure space factor $L^\infty(X) \rtimes \mathbb{F}_r$ has unique Cartan subalgebra, up to unitary conjugacy.

1. INTRODUCTION

See [OP] for the historical background. We assume every finite von Neumann algebra comes together with a distinguished faithful normal tracial state and every action on a finite von Neumann algebra is trace-preserving. A von Neumann algebra is said to be *diffuse* if it does not have a non-zero minimal projection. In this note, we state theorems and lemmas in general forms, but prove them only in the case of $Q = \mathbb{C}1$.

Theorem. *Let $\mathbb{F}_r \curvearrowright Q$ be an action of a free group on a finite von Neumann algebra. Assume $M = Q \rtimes \mathbb{F}_r$ has the CMAP. If $P \subset M$ is a diffuse amenable subalgebra and N denotes the von Neumann algebra generated by its normalizer $\mathcal{N}_M(P)$, then either N is amenable relative to Q inside M, or a non-zero corner of P can be conjugated into Q inside M.*

We mention three interesting applications of the Theorem, each corresponding to a particular choice of $\mathbb{F}_r \curvearrowright Q$. Thus, taking $Q = \mathbb{C}1$, we get:

Corollary 1. *The normalizer of any diffuse amenable subalgebra P of a free group factor $L(\mathbb{F}_r)$ generates an amenable von Neumann algebra.*

This strengthens two well known in-decomposability properties of free group factors: Voiculescu's result in [Vo], showing that $L(\mathbb{F}_r)$ has no Cartan subalgebras, and the author's result in [Oz] that the commutant in $L(\mathbb{F}_r)$ of any diffuse subalgebra must be amenable.

If we take Q to be an arbitrary finite factor with $\Lambda_{\mathrm{cb}}(Q) = 1$ and let $\mathbb{F}_r$ act trivially on it, then $M = Q \bar{\otimes} L(\mathbb{F}_r)$ has the CMAP and Theorem implies:

Corollary 2. *If Q is a II_1 factor with the CMAP then $Q \bar{\otimes} L(\mathbb{F}_r)$ does not have Cartan subalgebras.*

This shows in particular that any factor of the form $L(\mathbb{F}_r) \bar{\otimes} R$ or $L(\mathbb{F}_{r_1}) \bar{\otimes} L(\mathbb{F}_{r_2}) \bar{\otimes} \cdots$ does not have a Cartan subalgebra.

Problem. Get rid of the assumption that Q has the CMAP.

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Finally, if we take $\mathbb{F}_r \curvearrowright X$ to be a profinite measure preserving action on a probability measure space (X, μ), i.e. an action with the property that $L^\infty(X)$ is a limit of an increasing sequence of $\mathbb{F}_r$-invariant finite dimensional subalgebras Q_n of $L^\infty(X)$, then $N = L^\infty(X) \rtimes \mathbb{F}_r$ is an increasing limit of the algebras $Q_n \rtimes \mathbb{F}_r$, each one of which is an amplification of $L(\mathbb{F}_r)$. Since the CMAP behaves well to amplifications and inductive limits, it follows that N has the CMAP, so by applying Theorem and (A.1 in [Pol]) we get:

Corollary 3. *If $\mathbb{F}_r \curvearrowright X$ is a free ergodic measure preserving profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra of the* II_1*-factor $L^\infty(X) \rtimes \mathbb{F}_r$, up to unitary conjugacy.*

2. Preliminaries

2.1. **Finite von Neumann algebras.** We fix conventions for (semi-)finite von Neumann algebras, but before that we note that the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with I directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is *normal* if it is ultraweakly continuous. Whenever a *finite* von Neumann algebra M is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by τ. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state τ. If $M = L(\Gamma)$ is a group von Neumann algebra, then the tracial state τ is given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$ for $x \in L(\Gamma)$. Any von Neumann subalgebra $P \subset M$ is assumed to contain the unit of M and inherits the tracial state τ from M. The unique τ-preserving conditional expectation from M onto P is denoted by E_P. We denote by $\mathcal{Z}(M)$ the center of M; by $\mathcal{U}(M)$ the group of unitary elements in M; and by

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : (\mathrm{Ad}\, u)(P) = P\}$$

the normalizing group of P in M, where $(\mathrm{Ad}\, u)(x) = uxu^*$. A maximal abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{N}_M(A)'' = M$ is called a *Cartan subalgebra.* We note that if $\Gamma \curvearrowright X$ is an ergodic essentially-free probability-measure-preserving action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [FM].)

We refer the reader to the section IX.2 of [Ta] for the details of the following facts on noncommutative L^p-spaces. Let $\mathcal{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr. For $1 \le p < \infty$, we define the L^p-norm on $\mathcal{N}$ by $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$. By completing $\{x \in \mathcal{N} : \|x\|_p < \infty\}$ with respect to the L^p-norm, we obtain a Banach space $L^p(\mathcal{N})$. We only need $L^1(\mathcal{N})$, $L^2(\mathcal{N})$ and $L^\infty(\mathcal{N}) = \mathcal{N}$. The trace Tr extends to a contractive linear functional on $L^1(\mathcal{N})$. We occasionally write $\widehat{x}$ for $x \in \mathcal{N}$ when viewed as an element in $L^2(\mathcal{N})$. For any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$, there is a natural product map

$$L^p(\mathcal{N}) \times L^q(\mathcal{N}) \ni (x, y) \mapsto xy \in L^r(\mathcal{N})$$

which satisfies $\|xy\|_r \le \|x\|_p\|y\|_q$ for any x and y. The Banach space $L^1(\mathcal{N})$ is identified with the predual of $\mathcal{N}$ under the duality $L^1(\mathcal{N}) \times \mathcal{N} \ni (\zeta, x) \mapsto \mathrm{Tr}(\zeta x) \in \mathbb{C}$. The Banach space $L^2(\mathcal{N})$ is identified with the GNS-Hilbert space of $(\mathcal{N}, \mathrm{Tr})$. Elements in $L^p(\mathcal{N})$ can be regarded as closed operators on $L^2(\mathcal{N})$ which are affiliated with $\mathcal{N}$ and hence in addition to the above-mentioned product, there are well-defined notion of positivity, square root, etc. We will use many times the generalized Powers–Størmer inequality (Theorem XI.1.2 in [Ta]):

$$\|\eta - \zeta\|_2^2 \le \|\eta^2 - \zeta^2\|_1 \le \|\eta + \zeta\|_2\|\eta - \zeta\|_2 \tag{2.1}$$

for every $\eta, \zeta \in L^2(\mathcal{N})_+$. The Hilbert space $L^2(\mathcal{N})$ is an $\mathcal{N}$-bimodule such that $\langle x\xi y, \eta\rangle = \mathrm{Tr}(x\xi y\eta^*)$ for $\xi, \eta \in L^2(\mathcal{N})$ and $x, y \in \mathcal{N}$. We recall that this gives the canonical identification between the commutant $\mathcal{N}'$ of $\mathcal{N}$ in $\mathbb{B}(L^2(\mathcal{N}))$ and the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}} = \{x^{\mathrm{op}} : x \in \mathcal{N}\}$ of $\mathcal{N}$. Moreover, the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}}$ is $*$-isomorphic to the complex conjugate von Neumann algebra $\bar{\mathcal{N}} = \{\bar{x} : x \in \mathcal{N}\}$ of $\mathcal{N}$ under the $*$-isomorphism $x^{\mathrm{op}} \mapsto \bar{x}^*$.

Whenever $\mathcal{N}_0 \subset \mathcal{N}$ is a von Neumann subalgebra such that the restriction of Tr to $\mathcal{N}_0$ is still semi-finite, we identify $L^p(\mathcal{N}_0)$ with the corresponding subspace of $L^p(\mathcal{N})$. Anticipating a later use, we consider the tensor product von Neumann algebra $(\mathcal{N} \bar{\otimes} M, \mathrm{Tr} \otimes \tau)$ of a semi-finite von Neumann algebra $(\mathcal{N}, \mathrm{Tr})$ and a finite von Neumann algebra (M, τ). Then, $\mathcal{N} \cong \mathcal{N} \bar{\otimes} \mathbb{C}1 \subset \mathcal{N} \bar{\otimes} M$ and the restriction of $\mathrm{Tr} \otimes \tau$ to $\mathcal{N}$ is Tr. Moreover, the conditional expectation $\mathrm{id} \otimes \tau \colon \mathcal{N} \bar{\otimes} M \to \mathcal{N}$ extends to a contraction from $L^1(\mathcal{N} \bar{\otimes} M) \to L^1(\mathcal{N})$.

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation E_Q can be viewed as the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x) e_Q$ for every $x \in M$. The *basic construction* $\langle M, e_Q\rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_Q. We note that $\langle M, e_Q\rangle$ coincides with the commutant of the right Q-action in $\mathbb{B}(L^2(M))$. In particular, if $Q = \mathbb{C}1$, then $\langle M, \varepsilon_Q\rangle = \mathbb{B}(L^2(M))$. The linear span of $\{x e_Q y : x, y \in M\}$ is an ultraweakly dense $*$-subalgebra in $\langle M, e_Q\rangle$ and the basic construction $\langle M, e_Q\rangle$ comes together with the faithful normal semi-finite trace Tr such that $\mathrm{Tr}(x e_Q y) = \tau(xy)$. See Section 1.3 in [Po1] for more information on the basic construction.

2.2. Relative amenability. We adapt here Connes's characterization of amenable (injective) von Neumann algebras to the relative situation. Recall that for von Neumann algebras $N \subset \mathcal{N}$, a state φ on $\mathcal{N}$ is said to be *N-central* if $\varphi \circ \mathrm{Ad}(u) = \varphi$ for any $u \in \mathcal{U}(N)$, or equivalently if $\varphi(ax) = \varphi(xa)$ for all $a \in N$ and $x \in \mathcal{N}$.

Theorem 2.1. *Let Q and N be von Neumann subalgebras of a finite von Neumann algebra M. Then, the following are equivalent.*

1. *There exists an N-central state φ on $\langle M, e_Q\rangle$ such that $\varphi|_M = \tau$.*
2. *There exists a conditional expectation Φ from $\langle M, e_Q\rangle$ onto N such that $\Phi|_M = E_N$.*
3. *There exists a net (ξ_n) of unit vectors in $L^2\langle M, e_Q\rangle$ such that $\lim\langle x\xi_n, \xi_n\rangle = \tau(x)$ for every $x \in M$ and $\lim \|[u, \xi_n]\|_2 = 0$ for every $u \in N$.*

Definition 2.2. Let $Q, N \subset M$ be finite von Neumann algebras. We say N is *amenable relative to Q inside M* if any of the conditions in Theorem 2.1 holds.

We note that if N is amenable relative to an amenable von Neumann subalgebra Q, then N is amenable; and that for $M = Q \rtimes \Gamma$, the von Neumann subalgebra $L(\Gamma) \subset M$ is amenable relative to Q inside M iff Γ is amenable.

Problem. Let $Q, N \subset M$. Prove that N is amenable relative to Q inside M if and only if the following condition holds:

(4) There exists a conditional expectation Ψ from $\langle M, e_Q\rangle$ onto $N' \cap \langle M, e_Q\rangle$ such that $\Psi \circ \mathrm{Ad}(u) = \Psi$ for every $u \in \mathcal{U}(N)$.

2.3. Intertwining subalgebras inside II_1 factors. We extract from [Po1, Po2] some results which are needed later.

Theorem 2.3. *Let M be a finite von Neumann algebra and $P, Q \subset M$ be von Neumann subalgebras. Then, the following are equivalent.*

(1) There exists a non-zero projection $e \in \langle M, e_Q \rangle$ with $\mathrm{Tr}(e) < \infty$ such that the ultraweakly closed convex hull of $\{w^*ew : w \in \mathcal{U}(P)\}$ does not contain 0.

(2) There exist non-zero projections $p \in P$ and $q \in Q$, a normal $*$-homomorphism $\theta \colon pPp \to qQq$ and a non-zero partial isometry $v \in M$ such that

$$\forall x \in pPp \quad xv = v\theta(x)$$

and $v^*v \in \theta(pPp)' \cap qMq$, $vv^* \in p(P' \cap M)p$.

Definition 2.4. Let $P, Q \subset M$ be finite von Neumann algebras. We say that P *embeds into Q inside M* if any of the conditions in Theorem 2.3 holds.

Let $\langle M, e_Q \rangle$ be the basic construction of finite von Neumann algebras $Q \subset M$. We define $\mathbb{K}\langle M, e_Q \rangle$ to be the norm-closed linear span of $\{xe_Qy : x, y \in M\}$.

Corollary 2.5. *Let $P, Q \subset M$ be finite von Neumann algebras. Assume there exists a P-central state φ on $\langle M, e_Q \rangle$ which is normal on M and such that $\varphi(\mathbb{K}\langle M, e_Q \rangle) \neq \{0\}$. Then, P embeds into Q inside M.*

Proof in the case of $Q = \mathbb{C}1$. We restrict φ to $\mathbb{K}\langle M, e_Q \rangle = \mathbb{K}(L^2(M))$ and view it as the trace class operator h, i.e., $\varphi(x) = \mathrm{Tr}(hx)$ for $x \in \mathbb{K}(L^2(M))$. It follows that h is a non-zero compact operator which commutes with P. This implies P contains a non-zero minimal projection, i.e., P embeds into $Q = \mathbb{C}1$ inside M. Indeed, if P is diffuse, then there is a sequence (u_n) of unitary elements in P which converges to zero ultraweakly and $0 = \text{SOT-}\lim u_n h u_n^* = h$ for every compact operator h which commutes with P. □

Finally, recall that A.1 in [Po1] shows the following:

Lemma 2.6. *Let A and B be Cartan subalgebras of a type II_1-factor M. If A embeds into B inside M, then there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

2.4. **The complete metric approximation property.** Let Γ be a discrete group. For a function f on Γ, we write m_f for the multiplier on $\mathbb{C}\Gamma \subset L(\Gamma)$ defined by $m_f(g) = fg$ for $g \in \mathbb{C}\Gamma$. We simply write $\|f\|_{\mathrm{cb}}$ for $\|m_f\|_{\mathrm{cb}}$ and call it the Herz-Schur norm. We denote by

$$B_2(\Gamma) = \{f : \|f\|_{\mathrm{cb}} < \infty\}$$

the Banach space of Herz-Schur multipliers. Every $f \in B_2(\Gamma)$ (or precisely m_f) extends to a normal completely bounded map on $L(\Gamma)$ such that $m_f(\lambda(s)) = f(s)\lambda(s)$. We refer the reader to [BO] for an account of Herz-Schur multipliers.

Definition 2.7. A discrete group Γ is *weakly amenable* if there exist a constant $C \geq 1$ and a net (f_n) of finitely supported functions on Γ such that $\limsup \|f_n\|_{\mathrm{cb}} \leq C$ and $f_n \to 1$ pointwise. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ of Γ is defined as the infimum of the constant C for which a net (f_n) as above exists.

We say a finite von Neumann algebra M has the *(weak*) completely bounded approximation property* if there exist a constant $C \geq 1$ and a net (ϕ_n) of normal finite-rank maps on M such that $\limsup \|\phi_n\|_{\mathrm{cb}} \leq C$ and $\phi_n \to \mathrm{id}_M$ in the point-ultraweak topology. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(M)$ of M is defined as the infimum of the constant C for which a net (ϕ_n) as above exists. Also, we say that M has the *(weak*) complete metric approximation property (CMAP)* if $\Lambda_{\mathrm{cb}}(M) = 1$.

Haagerup proved that $\Lambda_{\rm cb}(M) = \Lambda_{\rm cb}(\Gamma)$ (the inequality $\leq$ is trivial: just take $\phi_n = m_{f_n}$). Thus, the following results imply the CMAP of $L(\mathbb{F}_r)$. For the following, we assume $r < \infty$ for simplicity.

Theorem 2.8. *Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $l(x)$ denote the canonical word length of $x \in \mathbb{F}_r$. Then, for every $z \in \mathbb{D}$, the function $\mathbb{F}_r \ni x \mapsto z^{l(x)} \in \mathbb{C}$ belongs to $B_2(\mathbb{F}_r)$ with*

$$\|z^l\|_{\rm cb} \leq \frac{|1-z|}{1-|z|}.$$

Moreover, $\mathbb{D} \ni z \mapsto z^l \in B_2(\mathbb{F}_r)$ is holomorphic and γ^l is positive definite for $\gamma \in \mathbb{R}$.

Proof. Every element $x \in \mathbb{F}_r = \langle g_1, \ldots, g_r \rangle$ is written uniquely as a reduced word $x = g_{k_1}^{\varepsilon_1} \cdots g_{k_n}^{\varepsilon_n}$, where $n \in \mathbb{N}_0$, $1 \leq k_i \leq n$ and $\varepsilon_n = \pm 1$ such that there is no consecutive $g_k^{-1} g_k$ nor $g_k g_k^{-1}$. The length $l(x)$ of the element x is n. We identify the free group $\mathbb{F}_r$ with its Cayley graph (w.r.t. the canonical generators), which is the $2r$-regular tree. The distance between $x, y \in \mathbb{F}_r$ is given by $d(x,y) = l(xy^{-1})$. (Warning: the choice $d(x,y) = l(x^{-1}y)$ is more common, but $d(x,y) = l(xy^{-1})$ is more compatible with the left regular representation.) A *geodesic path* in $\mathbb{F}_r$ is a finite or infinite sequence $x_0, x_1, \ldots$ of points in $\mathbb{F}_r$ such that $d(x_i, x_j) = |i-j|$ for all i and j.

We fix a point ω at infinity, i.e., ω is an infinite geodesic path (starting at the unit, say). For every $x \in \mathbb{F}_r$, there exists a unique geodesic path ω_x starting at x and eventually flows into ω, i.e., $\exists k \in \mathbb{Z}$ such that $\omega_x(i) = \omega(k+i)$ for sufficiently large i.

For $z \in \mathbb{D}$, we define $\zeta_z \in \ell^\infty(\mathbb{F}_r, \ell^2(\mathbb{F}_r))$ by

$$\zeta_z(x) = \sqrt{1-z^2} \sum_{i=0}^{\infty} z^i \delta_{\omega_x(i)},$$

where $\sqrt{1-z}$ is the principal branch of the square root. The series converges absolutely in z and the function $\mathbb{D} \ni z \mapsto \zeta_z \in \ell^\infty(\mathbb{F}_r, \ell^2(\mathbb{F}_r))$ is holomorphic. One has for every $x \in \mathbb{F}_r$ that

$$\|\zeta_z(x)\|^2 = |1-z^2| \sum_{i=0}^{\infty} |z|^{2i} = \frac{|1-z^2|}{1-|z|^2} \leq \frac{|1-z|}{1-|z|}$$

and that

$$\begin{aligned} \langle \zeta_z(y), \overline{\zeta_z(x)} \rangle &= (1-z^2) \sum_{i,j=1}^{\infty} z^{i+j} \delta_{\omega_x(i), \omega_y(j)} \\ &= (1-z^2) \sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} \delta_{\omega_x(i), \omega_y(n-i)} \end{aligned}$$

now we observe that for every n one has $\omega_x(i) = \omega_y(n-i)$ for at most one i and $n - d(x,y) \in 2\mathbb{N}_0$; and hence

$$\begin{aligned} &= (1-z^2) \sum_{m=0}^{\infty} z^{d(x,y)+2m} \\ &= z^{d(x,y)}. \end{aligned}$$

We define $V_z, W_z \in \mathbb{B}(\ell^2(\mathbb{F}_r), \ell^2(F_r) \otimes \ell^2(\mathbb{F}_r))$ by

$$V_z \delta_x = \delta_x \otimes \overline{\zeta_z(x)} \text{ and } W_z \delta_y = \delta_y \otimes \zeta_z(y).$$

It follows that

$$\langle V_z^*(\lambda(s)\otimes 1)W_z\delta_y,\delta_x\rangle = \langle \delta_{sy},\delta_x\rangle\langle \zeta_z(y),\overline{\zeta_z(x)}\rangle = \langle \lambda(s)\delta_y,\delta_x\rangle z^{l(s)},$$

which implies $V_z^*(\lambda(s)\otimes 1)W_z = m_{z^l}(\lambda(s))$. Moreover, if $z\in\mathbb{R}$, then $V_z = W_z$ and m_{z^l} is u.c.p. Since

$$\|V_z\|^2 = \|W_z\|^2 = \|\zeta_z\|^2_{\ell^\infty(\mathbb{F}_r,\ell^2(\mathbb{F}_r))} \le \frac{|1-z|}{1-|z|},$$

we are done. □

Theorem 2.9 (De Cannière and Haagerup). $\Lambda_{\mathrm{cb}}(\mathbb{F}_r)=1$.

Proof. Since $\|z^t\|_{\mathrm{cb}}=1$ for $t\in(0,1)$ and $z^t\to 1$ pointwise as $t\to 1$, it suffices to show z^t can be approximated in $B_2(\mathbb{F}_r)$ by finitely supported functions. Let $B_2^0\subset B_2(\mathbb{F}_r)$ be the norm-closure of the finitely supported functions. Since $z^l\in\ell^1(\mathbb{F}_r)$ for $|z|<(2r-1)^{-1}$, one has $z^l\in B_2^0$ for $|z|<(2r-1)^{-1}$. The function $\mathbb{D}\ni z\mapsto z^l\in B_2(\mathbb{F}_r)/B_2^0$ is holomorphic on $\mathbb{D}$ and zero for $|z|<(2r-1)^{-1}$. Hence, by uniqueness of holomorphic extensions, it is zero everywhere. □

3. Weakly compact actions

For a finite von Neumann algebra P, let J be the conjugate unitary on $L^2(P)$ defined by $J\hat{x}=\widehat{x^*}$. Then, we have $P'=JPJ$ and P' is $*$-isomorphic to the complex-conjugate von Neumann algebra $\bar{P}=\{\bar{x}: x\in P\}$ via $JxJ\mapsto\bar{x}$.

Definition 3.1. Let σ be an action of a group Γ on a finite von Neumann algebra P. We say the action σ is *profinite* if there exists an increasing net (P_n) of Γ-invariant finite-dimensional von Neumann subalgebras of P such that $P=(\bigcup P_n)''$. We say the action σ is *weakly compact* if there exists a net (η_n) of unit vectors in $L^2(P\bar{\otimes}\bar{P})_+$ such that

- $\|\eta_n-(v\otimes\bar{v})\eta_n\|_2\to 0$ for every $v\in\mathcal{U}(P)$.
- $\|\eta_n-(\sigma_g\otimes\bar{\sigma}_g)(\eta_n)\|_2\to 0$ for every $g\in\Gamma$.
- $(\tau\otimes\mathrm{id})(\eta_n^2)=1=(\mathrm{id}\otimes\tau)(\eta_n^2)$ for every n.

Here, we identify σ as the corresponding unitary representation on $L^2(P)$.

Proposition 3.2. *Let σ be an action of a group Γ on a finite von Neumann algebra P and consider the following conditions.*

(1) The action σ is profinite.
(2) The action σ is compact and the von Neumann algebra P is amenable.
(3) there exists a net (μ_n) of normal states on $P\bar{\otimes}\bar{P}$ such that
- $\mu_n(v\otimes\bar{v})\to 1$ *for every* $v\in\mathcal{U}(P)$.
- $\|\mu_n-\mu_n\circ(\sigma_g\otimes\bar{\sigma}_g)\|\to 0$ *for every* $g\in\Gamma$.
- $\mu_n(x\otimes 1)=\tau(x)=\mu_n(1\otimes\bar{x}^*)$ *for every n and $x\in P$.*

(4) The action σ is weakly compact.
(5) There exists a state φ on $\mathbb{B}(L^2(P))$ such that $\varphi|_P=\tau$ and $\varphi\circ\mathrm{Ad}\,u=\varphi$ for all $u\in\mathcal{U}(P)\cup\sigma(\Gamma)$.

Then, one has (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

We only prove (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Rightarrow$ (3): Suppose that σ is profinite and take a net (P_n) as in the definition. Let n be fixed. We note that the τ-preserving conditional expectation E_n from P onto P_n is Γ-equivariant: $E_n \circ \mathrm{Ad}(u) = \mathrm{Ad}(u) \circ E_n$ for every $u \in \sigma(\Gamma)$. We define a state on the algebraic tensor product $P \otimes \bar{P}$ by

$$\mu_n(\sum_k a_k \otimes \bar{b}_k) = \sum_k \tau(E_n(a_k)b_k^*) = \langle E_n(a_k)\hat{1}b_k^*, \hat{1}\rangle_{L^2(P)}.$$

Since P_n is finite-dimensional, μ_n is contractive w.r.t. the minimal tensor norm and moreover μ_n extends to a normal state on $P \bar{\otimes} \bar{P}$. It is not difficult to see that (μ_n) satisfies the conditions in (3).

(3) $\Leftrightarrow$ (4): This follows from the Powers-Størmer inequality (2.1) and the inequality (3.1) below.

(4) $\Rightarrow$ (5): The state φ on $\mathbb{B}(L^2(P))$, defined by

$$\varphi(x) = \mathrm{Lim}\langle (x \otimes 1)\eta_n, \eta_n\rangle,$$

satisfies the condition (5). Indeed,

$$(\varphi \circ \mathrm{Ad}(u))(x) = \mathrm{Lim}\langle (x \otimes 1)(u \otimes \bar{u})\eta_n, (u \otimes \bar{u})\eta_n\rangle = \varphi(x)$$

for every $x \in \mathbb{B}(L^2(P))$ and $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. □

The following is the main theorem of this section.

Theorem 3.3. *Let M be a finite von Neumann algebra with the CMAP and $P \subset M$ be a von Neumann subalgebra. Then, the conjugation action of $\mathcal{N}_M(P)$ on P is weakly compact.*

We need the following consequence of Connes's theorem (Theorem 2.1).

Lemma 3.4. *Let M be a finite von Neumann algebra, $P \subset M$ be an amenable von Neumann subalgebra and $u \in \mathcal{N}_M(P)$. Then, the von Neumann algebra Q generated by P and u is amenable.*

Proof. Since P is injective, the τ-preserving conditional expectation E_P from M onto P extends to a u.c.p. map $\tilde{E}_P$ from $\mathbb{B}(L^2(M))$ onto P. We note that $\tilde{E}_P$ is a conditional expectation: $\tilde{E}_P(axb) = a\tilde{E}_P(x)b$ for every $a, b \in P$ and $x \in \mathbb{B}(L^2(M))$. We define a state φ on $\mathbb{B}(L^2(M))$ by

$$\varphi(x) = \mathrm{Lim}_n \frac{1}{n}\sum_{k=0}^{n-1} \tau(\tilde{E}_P(u^k x u^{-k})).$$

It is not hard to check that $\varphi|_M = \tau$, $\varphi \circ \mathrm{Ad}(u) = \varphi$ and $\varphi \circ \mathrm{Ad}(v) = \varphi$ for every $v \in \mathcal{U}(P)$. It follows that φ is a Q-central state with $\varphi|_Q = \tau$. By Connes's theorem, this implies that Q is amenable. □

Proof of Theorem 3.3. First we note the following general fact: Let ω be a state on a C*-algebra N and $u \in \mathcal{U}(N)$. We define $\omega_u(x) = \omega(xu^*)$ for $x \in N$. Then, one has

$$\max\{\|\omega - \omega_u\|, \|\omega - \omega \circ \mathrm{Ad}(u)\|\} \leq 2\sqrt{2|1 - \omega(u)|}. \tag{3.1}$$

Indeed, one has $\|\xi_\omega - u^*\xi_\omega\|^2 = 2(1 - \Re\omega(u)) \leq 2|1 - \omega(u)|$, where ξ_ω is the GNS-vector for ω.

Let (ϕ_n) be a net of normal finite rank maps on M such that $\limsup \|\phi_n\|_{\mathrm{cb}} \leq 1$ and $\|x - \phi_n(x)\|_2 \to 0$ for all $x \in M$. We observe that the net $(\tau \circ \phi_n)$ converges to τ weakly in M_*. Hence by the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that $\|\tau - \tau\circ\phi_n\| \to 0$. Let μ be the $*$-representation of the algebraic tensor product $M \otimes \bar{M}$ on $L^2(M)$ defined by

$$\mu(\sum_k a_k \otimes \bar{b}_k)\xi = \sum_k a_k \xi b_k^*.$$

We define a linear functional μ_n on $M \otimes \bar{M}$ by

$$\mu_n(\sum_k a_k \otimes \bar{b}_k) = \langle \mu(\sum_k \phi_n(a_k) \otimes \bar{b}_k)\hat{1}, \hat{1}\rangle_{L^2(M)} = \tau(\sum_k \phi_n(a_k) b_k^*).$$

Since ϕ_n is normal and of finite rank, μ_n extends to a normal linear functional on $M \bar{\otimes} \bar{M}$, which is still denoted by μ_n. For an amenable von Neumann subalgebra $Q \subset M$, we denote by μ_n^Q the restriction of μ_n to $Q \bar{\otimes} \bar{Q}$. Since Q is amenable, the $*$-representation μ is continuous with respect to the spatial tensor norm on $Q \otimes \bar{Q}$ and hence $\|\mu_n^Q\| \le \|\phi_n\|_{\mathrm{cb}}$. We denote $\omega_n^Q = \|\mu_n^Q\|^{-1}|\mu_n^Q|$. Since $\limsup \|\mu_n^Q\| \le 1$ and $\lim \mu_n^Q(1\otimes 1) = 1$, the inequality (3.1), applied to ω_n^Q, implies that

$$\limsup_n \|\mu_n^Q - \omega_n^Q\| = 0. \tag{3.2}$$

Now, consider the case $Q = P$. By (3.2), one has

$$\lim_n \omega_n^P(v \otimes \bar{v}) = \lim_n \mu_n^P(v \otimes \bar{v}) = \lim_n \tau(\phi_n(v)v^*) = 1 \tag{3.3}$$

for any $v \in \mathcal{U}(P)$. Now, let $u \in \mathcal{N}(P)$ and consider the case $Q = \langle P, u\rangle$, which is amenable by Lemma 3.4. Since $\mu_n^{\langle P,u\rangle}(u \otimes \bar{u}) = \tau(\phi_n(u)u^*) \to 1$, one has

$$\limsup_n \|\mu_n^{\langle P,u\rangle} - \mu_n^{\langle P,u\rangle} \circ \mathrm{Ad}(u \otimes \bar{u})\| = 0 \tag{3.4}$$

by (3.1) and (3.2). But since $(\mu_n^{\langle P,u\rangle} \circ \mathrm{Ad}(u \otimes \bar{u}))|_{P\bar{\otimes}\bar{P}} = \mu_n^P \circ \mathrm{Ad}(u \otimes \bar{u})$, one has

$$\limsup_n \|\omega_n^P - \omega_n^P \circ \mathrm{Ad}(u \otimes \bar{u})\| = 0 \tag{3.5}$$

by (3.2) and (3.4). Therefore, (ω_n^P) satisfies Proposition 3.2.(3). □

4. Main Results

Theorem 4.1. *Let $M = Q \rtimes \mathbb{F}_r$ be the crossed product of a finite von Neumann algebra Q and the free group $\mathbb{F}_r$ of rank $2 \le r \le \infty$ acting on Q (need not be ergodic nor free). Let $P \subset M$ and assume that the conjugation action of $\mathcal{N}_N(P)$ on P is weakly compact. (This is automatic if M has the CMAP.) Then, either P embeds into Q inside M, or $\mathcal{N}_M(P)''$ is amenable relative to Q inside M.*

For the proof of Theorem 4.1, recall from [Po3] the construction of 1-parameter automorphisms α_t ("malleable deformation") of $L(\mathbb{F}_r * \tilde{\mathbb{F}}_r)$, where $\tilde{\mathbb{F}}_r$ is a copy of $\mathbb{F}_r$. Let $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) be the standard generators of $\mathbb{F}_r$ (resp. $\tilde{\mathbb{F}}_r$) viewed as unitary elements in $L(\mathbb{F}_r * \tilde{\mathbb{F}}_r)$. Let $b_k^t = \exp(t \log b_k)$, where where log is the principal branch of the complex logarithm ($\sqrt{-1}\log z \in (-\pi, \pi]$ for $z \in \mathbb{C}$ with $|z| = 1$). The $*$-automorphism α_t is defined by $\alpha_t(a_k) = a_k b_k^t$ and $\alpha_t(b_k) = b_k$.

In this paper, we adapt this construction to $\mathbb{F}_r \curvearrowright Q$ and $M = Q \rtimes \mathbb{F}_r$. We extend the action $\mathbb{F}_r \curvearrowright Q$ to that of $\mathbb{F}_r * \tilde{\mathbb{F}}_r$, by letting $\tilde{\mathbb{F}}_r$ act trivially on Q. We consider

$$\tilde{M} = Q \rtimes (\mathbb{F}_r * \tilde{\mathbb{F}}_r) = M *_Q (Q \bar{\otimes} L(\tilde{\mathbb{F}}_r))$$

and redefine the $*$-homomorphism $\alpha_t\colon M \to \tilde{M}$ by $\alpha_t(x) = x$ for $x \in Q$ and $\alpha_t(a_k) = a_k b_k^t$ for each k. (We can define α_t on $\tilde{M}$, but we do not need it.)

Let

$$\gamma(t) = \tau(b_k^t) = \frac{1}{2}\int_{-1}^{1} \exp(t\pi\sqrt{-1}s)\,ds = \frac{\sin(t\pi)}{t\pi}$$

and $\phi_{\gamma(t)}\colon L(\mathbb{F}_r) \to L(\mathbb{F}_r)$ be the Haagerup multiplier (Theorem 2.8) associated with the positive type function $g \mapsto \gamma(t)^{l(g)}$ on $\mathbb{F}_r$. We may extend $\phi_{\gamma(t)}$ to M by defining $\phi_{\gamma(t)}(x\lambda(g)) = x\phi_{\gamma(t)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\mathbb{F}_r)$. We relate α_t and $\phi_{\gamma(t)}$ as follows.

Lemma 4.2. *One has $E_M \circ \alpha_t = \phi_{\gamma(t)}$.*

Proof by Example. Let $Q = \mathbb{C}1$ and $x = a_1 a_1 a_2^{-1}$. Then, $\alpha_t(x) = a_1 b_1^t a_1 b_1^t b_2^{-t} a_2^{-1}$. Since the von Neumann algebras $W^*(a_1), \ldots, W^*(b_1), \ldots$ are mutually free, one has

$$(E_M \circ \alpha_t)(x) = a_1\tau(b_1^t)a_1\tau(b_1^t b_2^{-t})a_2^{-1} = \tau(b_1^t)\tau(b_1^t)\tau(b_2^{-t})a_1 a_1 a_2^{-1} = \gamma(t)^3 x.$$

□

In particular, the τ-preserving u.c.p. map $E_N \circ \alpha_t$ on M is compact as an operator on $L^2(M)$. (Assume $r < \infty$ for simplicity.)

Let $Q \subset M \subset \tilde{M}$ be as above, and consider the basic construction $\langle M, e_Q\rangle$ of $(Q \subset M)$. Then, $L^2\langle M, e_Q\rangle$ is naturally an M-bimodule.

Lemma 4.3. *Let $Q \subset M \subset \tilde{M}$ be as above. Then, $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic as an M-bimodule to a multiple of $L^2\langle M, e_Q\rangle$.*

Proof in the case of $Q = \mathbb{C}1$. Let X be the subset of $\mathbb{F}_r * \tilde{\mathbb{F}}_r$ consisting those elements whose initial and last letters in the reduced forms come from $\tilde{\mathbb{F}}_r$. It follows that every element of $\mathbb{F}_r * \tilde{\mathbb{F}}_r \setminus \mathbb{F}_r$ is uniquely written as sxt, where $s, t \in \mathbb{F}_r$ and $x \in X$. Now, one has

$$L^2(\tilde{M}) \ominus L^2(M) = \ell^2(\mathbb{F}_r * \tilde{\mathbb{F}}_r \setminus \mathbb{F}_r) \cong \ell^2(\mathbb{F}_r) \otimes \ell^2(X) \otimes \ell^2(\mathbb{F}_r)$$

as $L(\mathbb{F}_r)$-bimodule. □

In particular, when $Q = \mathbb{C}1$ and $M = L(\mathbb{F}_r)$, the representation of $L(\mathbb{F}_r) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$ on $L^2(\tilde{M}) \ominus L^2(M)$, defined by

$$(a \otimes b^{\mathrm{op}})(\xi) = a\xi b$$

for $a, b \in L(\mathbb{F}_r)$ and $\xi \in L^2(\tilde{M}) \ominus L^2(M)$, naturally extends to a representation of $\mathbb{B}(\ell^2(\mathbb{F}_r)) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$.

Proof of Theorem 4.1 in the case $Q = \mathbb{C}1$. Let $M = L(\mathbb{F}_r)$ and $P \subset M$ be a diffuse amenable von Neumann subalgebra. We will prove that $\mathcal{N}_M(P)''$ is amenable. Let a finite subset $F \subset \mathcal{N}_M(P)$ and $\varepsilon > 0$ be given.

We choose and fix $t > 0$ such that $\alpha = \alpha_t$ satisfies $\|u - \alpha(u)\|_2 < \varepsilon/4$ for every $u \in F$. Let (η_n) be the net of unit vectors in $L^2(P \bar{\otimes} \bar{P})_+$ satisfying the conditions in Definition 3.1. Let v_α be α viewed as an isometry from $L^2(M)$ into $L^2(\tilde{M})$ and consider $\eta_n^\alpha = (v_\alpha \otimes 1)\eta_n$. We note that

$$\langle (x \otimes 1)\eta_n^\alpha, \eta_n^\alpha\rangle = \tau(\alpha^{-1}(E_{\alpha(M)}(x))) = \tau(x) \tag{4.1}$$

for every n and $x \in \tilde{M}$. It follows that

$$\limsup_n \|[u \otimes \bar{u}, \eta_n^\alpha]\|_2 < \varepsilon/2 + \limsup_n \|[\alpha(u) \otimes \bar{u}, \eta_n^\alpha]\|_2 = \varepsilon/2 \tag{4.2}$$

for every $u \in F$. Let e_M be the orthogonal projection from $L^2(\tilde{M})$ onto $L^2(M)$ and

$$\zeta_n = ((1 - e_M) \otimes 1)\eta_n^\alpha \in \left(L^2(\tilde{M}) \ominus L^2(M)\right) \bar{\otimes} L^2(\bar{P}).$$

We note $T = e_M v_\alpha \in \mathbb{K}(L^2(M))$, by Lemma 4.2. Since η_n is approximately P-central, Corollary 2.5 implies

$$\lim_n \|\eta_n^\alpha - \zeta_n\|_2 = \lim_n \|(T \otimes 1)\eta_n\|_2 = 0. \tag{4.3}$$

By Lemma 4.3, the representation σ of $L(\mathbb{F}_r) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$ on $L^2(\tilde{M}) \ominus L^2(M)$ naturally extends to $\mathbb{B}(\ell^2(\mathbb{F}_r)) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$. Now, we define a state $\varphi_{F,\varepsilon}$ on $\mathbb{B}(\ell^2(\mathbb{F}_r))$ by

$$\varphi_{F,\varepsilon}(x) = \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)\zeta_n, \zeta_n \rangle$$

We note that if $x \in L(\mathbb{F}_r)$, then (4.1) and (4.3) imply

$$\varphi_{F,\varepsilon}(x) = \operatorname*{Lim}_{n\to\infty} \langle (x \otimes 1)\zeta_n, \zeta_n \rangle = \tau(x).$$

Moreover, if $u \in F$, then (4.2) and (4.3) imply

$$\begin{aligned}
\varphi_{F,\varepsilon}(u^*xu) &= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(u^*xu \otimes 1) \otimes \bar{u}^*\bar{u})\zeta_n, \zeta_n \rangle \\
&= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)(u \otimes \bar{u})\zeta_n, (u \otimes \bar{u})\zeta_n \rangle \\
&\approx_\varepsilon \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)\big(\zeta_n(u \otimes \bar{u})\big), \big(\zeta_n(u \otimes \bar{u})\big) \rangle \\
&= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes (u^{\mathrm{op}})^* u^{\mathrm{op}}) \otimes \bar{u}^*\bar{u})\zeta_n, \zeta_n \rangle \\
&= \varphi_{F,\varepsilon}(x)
\end{aligned}$$

for all contractions $x \in \mathbb{B}((\ell^2(\mathbb{F}_r))$. It follows that the state φ on $\mathbb{B}((\ell^2(\mathbb{F}_r))$, defined by

$$\varphi(x) = \operatorname*{Lim}_{F,\varepsilon} \varphi_{F,\varepsilon}(x),$$

satisfies $\varphi|_M = \tau$ and $\varphi \circ \mathrm{Ad}(u) = \varphi$ for all $u \in \mathcal{N}_M(P)$. It follows that $\varphi(ax) = \varphi(xa)$ for all a in the linear span of $\mathcal{N}_M(P)$ and $x \in \mathbb{B}(\ell^2(\mathbb{F}_r))$. By Cauchy-Schwarz inequality, this implies that $\varphi(ax) = \varphi(xa)$ for all $a \in \mathcal{N}_M(P)''$ and $x \in \mathbb{B}(\ell^2(\mathbb{F}_r))$, i.e., φ is an $\mathcal{N}_M(P)''$-central state such that $\varphi|_{\mathcal{N}_M(P)''} = \tau$. This implies that $\mathcal{N}_M(P)''$ is amenable (Theorem 2.1). □

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5. (おまけ) Gaboriau's theorem after Lück, Sauer and Thom

5.1. Background in homological algebra. Throughout this section, R is a unital ring and V is a left R-module.

Definition 5.1. A *complex* $\mathbf{V}$ consists of sequences of modules and morphisms

$$\mathbf{V}: \quad \cdots \longrightarrow V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \longrightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n. The n-th homology module of $\mathbf{V}$ is defined to be $H_n(\mathbf{V}) = \ker \partial_n / \operatorname{ran} \partial_{n+1}$. The complex $\mathbf{V}$ is *exact* if $H_n(\mathbf{V}) = 0$ for all n.

A *morphism* $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ consists of a sequence of morphisms $\varphi_n\colon V_n \to W_n$ such that $\varphi_n \circ \partial_{n+1} = \partial'_{n+1} \circ \varphi_{n+1}$ for all n. Since $\varphi_n(\operatorname{ran}\partial_{n+1}) \subset \operatorname{ran}\partial'_{n+1}$ and $\varphi_n(\ker\partial_{n+1}) \subset \ker\partial'_{n+1}$, the morphism $\boldsymbol{\varphi}$ induces morphisms $\varphi_{*,n}\colon H_n(\mathbf{V}) \to H_n(\mathbf{W})$.

A morphism $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ is *null-homotopic* if there is a sequence of morphisms $h_n\colon V_n \to W_{n+1}$ such that $\varphi_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$:

$$\begin{tikzcd}
\cdots \arrow[r] & V_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\varphi_{n+1}"] \arrow[dl] & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] \arrow[dl, "h_n"'] & V_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & W_{n+1} \arrow[r, "\partial'_{n+1}"] & W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Morphisms $\boldsymbol{\varphi}$, $\boldsymbol{\psi}\colon \mathbf{V} \to \mathbf{W}$ are *homotopic* if $\boldsymbol{\varphi} - \boldsymbol{\psi}$ is null-homotopic.

Lemma 5.2. *If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are homotopic, then $\varphi_{*,n} = \psi_{*,n}$ for all n.*

Proof. If $\boldsymbol{\varphi}$ is null-homotopic, then $\varphi_n(\ker\partial_n) = (\partial'_{n+1} \circ h_n)(\ker\partial_n) \subset \operatorname{ran}\partial'_{n+1}$ and hence $\varphi_{*,n} = 0$. The general case follows from this. □

Theorem 5.3. *Let complexes $\mathbf{V}$, $\mathbf{W}$ and a morphism $\varphi\colon V \to W$ be given*

$$\begin{tikzcd}
\mathbf{V}: & \cdots \arrow[r] & V_n \arrow[r, "\partial_n"] & V_{n-1} \arrow[r] & \cdots \arrow[r] & V_0 \arrow[r, two heads] & V \arrow[d, "\varphi"] \\
\mathbf{W}: & \cdots \arrow[r] & W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r] & \cdots \arrow[r] & W_0 \arrow[r, two heads] & W
\end{tikzcd}$$

such that every V_n ($n \geq 0$) is projective and $\mathbf{W}$ is exact. Then, there exists a morphism $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ which extends φ. Moreover, the extension $\boldsymbol{\varphi}$ is unique up to homotopy.

Proof. (Existence.) We proceed by induction. Let $\varphi_{-1} = \varphi$ and $\varphi_{-2} = 0$, and suppose we have constructed $\varphi_{-2}, \cdots, \varphi_{n-1}$ satisfying $\varphi_{m-2} \circ \partial_{m-1} = \partial'_{m-1} \circ \varphi_{m-1}$ for $m \leq n$:

$$\begin{tikzcd}
V_n \arrow[r, "\partial_n"] \arrow[d, dashed] & V_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\varphi_{n-1}"] & V_{n-2} \arrow[d, "\varphi_{n-2}"] \\
W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r, "\partial'_{n-1}"] & W_{n-2}
\end{tikzcd}$$

Since $\partial'_{n-1} \circ \varphi_{n-1} \circ \partial_n = \varphi_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$, one has $\operatorname{ran}\varphi_{n-1} \circ \partial_n \subset \operatorname{ran}\partial'_n$ by exactness. Since V_n is projective, there is a morphism $\varphi_n\colon V_n \to W_n$ which lifts $\varphi_{n-1} \circ \partial_n$ through ∂'_n, i.e., $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n$.

(Uniqueness.) It suffices to show that any extension φ of $\varphi = 0$ is null-homotopic. Let $h_{-1} = 0$ and $h_{-2} = 0$, and suppose we have constructed $h_{-2}, \ldots, h_{n-1}$ satisfying $\varphi_{m-1} = \partial'_m \circ h_{m-1} + h_{m-2} \circ \partial_{m-1}$ for $m \leq n$:

$$\begin{tikzcd}
 & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] \arrow[dl, dashed] & V_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\varphi_{n-1}"] \arrow[dl, "h_{n-1}"'] & V_{n-2} \arrow[dl, "h_{n-2}"'] \\
W_{n+1} \arrow[r, "\partial'_{n+1}"] & W_n \arrow[r, "\partial'_n"] & W_{n-1} &
\end{tikzcd}$$

Since $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n = (\partial'_n \circ h_{n-1} + h_{n-2} \circ \partial_{n-1}) \circ \partial_n = \partial'_n \circ h_{n-1} \circ \partial_n$, one has $\operatorname{ran}(\varphi_n - h_{n-1} \circ \partial_n) \subset \operatorname{ran} \partial'_{n+1}$ by exactness. Since V_n is projective, there is a morphism $h_n \colon V_n \to W_{n+1}$ such that $\partial'_{n+1} \circ h_n = \varphi_n - h_{n-1} \circ \partial_n$. □

Definition 5.4. For a module V, a *projective resolution* of V is an exact complex

$$\mathbb{V}: \qquad \cdots \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} V \longrightarrow 0$$

with all V_n $(n \geq 0)$ projective.

Definition 5.5. For a right R-module M and a left R-module V, define

$$\operatorname{Tor}_n^R(M, V) = H_n(M \otimes_R \mathbb{V}_{\geq 0}),$$

where $\mathbb{V}$ is any projective resolution of V and $M \otimes_R \mathbb{V}_{\geq 0}$ is the complex

$$M \otimes_R \mathbb{V}_{\geq 0}: \qquad \cdots \longrightarrow M \otimes_R V_n \longrightarrow \cdots \longrightarrow M \otimes_R V_1 \xrightarrow{\partial_1} M \otimes_R V_0 \longrightarrow 0.$$

Note that $M \otimes_R \mathbb{V}_{\geq 0}$ is given by omitting the term $M \otimes_R V$ from $M \otimes_R \mathbb{V}$.

Remark 5.6. Every module V has a projective (or even free) resolution, and the projective resolution is unique up to homotopy. It follows that the complex $M \otimes_R \mathbb{V}_{\geq 0}$ used to define $\operatorname{Tor}_\bullet^R(M, V)$ is also unique up to homotopy and hence $\operatorname{Tor}_\bullet^R(M, V)$ does not depends on the choice of a projective resolution of V.

We recall that the relative tensor product $M \otimes_R V$ is defined to be the $\mathbb{Z}$-module generated by $\{a \otimes \xi : a \in M,\ \xi \in V\}$ and factored out by the relations $a \otimes \xi + b \otimes \xi - (a + b) \otimes \xi$, $a \otimes \xi + a \otimes \eta - a \otimes (\xi + \eta)$, and $ar \otimes \xi - a \otimes r\xi$. If M is an S-R-module, then $M \otimes_R V$ is naturally a left S-module. We note that the relative tensor product operation $\otimes_R$ is associative and distributive w.r.t. a direct sum.

Examples. $M \otimes_R R = M$ and $R \otimes_R V = V$.

The module $\operatorname{Tor}_n^R(M, V)$ can be non-zero because $M \otimes_R \cdot$ needs not be a short exact functor. Namely, $V_2 \rightarrowtail V_1$ does not imply $M \otimes_R V_2 \rightarrowtail M \otimes_R V_1$. (The symbol $\rightarrowtail$ is used for injection.) However the functor $M \otimes_R \cdot$ is always right exact.

Lemma 5.7 (Right exactness). *Let M be arbitrary. If $V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \to 0$ is exact, then $M \otimes_R V_2 \xrightarrow{\mathrm{id} \otimes \partial_2} M \otimes_R V_1 \xrightarrow{\mathrm{id} \otimes \partial_1} M \otimes_R V_0 \to 0$ is exact.*

Proof. Exactness at $M \otimes_R V_0$ is clear. Since $(\mathrm{id} \otimes \partial_1) \circ (\mathrm{id} \otimes \partial_2) = \mathrm{id} \otimes (\partial_1 \circ \partial_2) = 0$, the morphism $\mathrm{id} \otimes \partial_1$ induces a morphism $\tilde{\partial}_1 \colon M \otimes_R V_1 / \operatorname{ran}(\mathrm{id} \otimes \partial_2) \twoheadrightarrow M \otimes_R V_0$. It is left to show that $\tilde{\partial}_1$ is injective. For this, it suffices to construct the left inverse σ of $\tilde{\partial}_1$: For $\sum a_i \otimes \xi_i \in M \otimes_R V_0$, define $\sigma(\sum a_i \otimes \xi_i) = \sum a_i \otimes \tilde{\xi}_i + \operatorname{ran}(\mathrm{id} \otimes \partial_2)$, where $\tilde{\xi}_i \in V_1$ is any lift of ξ_i. Then, σ is a well-defined morphism with $\sigma \circ \tilde{\partial}_1 = \mathrm{id}$. □

Definition 5.8. A right S-module N is *flat* if $N \otimes_S \cdot$ is an exact functor.

Note that free modules and projective modules are flat.

Lemma 5.9. *For a right S-module N, the following are equivalent.*

(1) N is flat.
(2) $\ker(\mathrm{id} \otimes \varphi) = N \otimes_S \ker\varphi$ for any morphism $\varphi\colon W \to V$.
(3) $H_\bullet(N \otimes_S \mathbb{V}) = N \otimes_S H_\bullet(\mathbb{V})$ for any complex $\mathbb{V}$ of S-modules.
(4) $N \otimes_S V \rightarrowtail N \otimes_S F$ for every f.g. modules $V \subset F$ with F free.

In particular, if N is flat, then for any S-R-module M and any left R-module V,

$$N \otimes_S \mathrm{Tor}^R_\bullet(M, V) = \mathrm{Tor}^R_\bullet(N \otimes_S M, V).$$

Proof. It is routine to check the equivalence of the conditions (1)–(3). (Use right exactness.) We only prove the implication (4)$\Rightarrow$(1). We first observe that the f.g. assumption on V and F can be dropped by continuity of a tensor product w.r.t. inductive limits. Let $\iota\colon W_1 \hookrightarrow W_2$ be given. We will show $N \otimes_S W_1 \rightarrowtail N \otimes_S W_2$. Take a free S-module F and a surjection $\pi\colon F \twoheadrightarrow W_2$, and set $V = \ker\pi$. Then, we have a commuting diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & \ker(\mathrm{id} \otimes \iota) \arrow[d] & \\
 & N \otimes_S V \arrow[r] \arrow[d, equal] & N \otimes_S \pi^{-1}(W_1) \arrow[r] \arrow[d] & N \otimes_S W_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes_S V \arrow[r] \arrow[d] & N \otimes_S F \arrow[r] & N \otimes_S W_2 & \\
 & 0 & & &
\end{tikzcd}$$

which is exact everywhere. By Snake Lemma, one has $\ker(\mathrm{id} \otimes \iota) = 0$. □

For the later purpose, we need the following. A (full) subcategory $\mathcal{D}$ of modules is a *Serre subcategory* if for every short exact sequence $0 \to V_2 \to V_1 \to V_0 \to 0$, one has $V_1 \in \mathcal{D} \Leftrightarrow V_0, V_2 \in \mathcal{D}$. A morphism $\varphi\colon V \to W$ is an *isomorphism modulo $\mathcal{D}$* if both $\ker\varphi$ and $\operatorname{coker}\varphi = V/\operatorname{ran}\varphi$ are in $\mathcal{D}$.

Lemma 5.10. *Let $\mathcal{D}$ be a Serre subcategory. Let $\mathbb{V}$ and $\mathbb{W}$ be complexes of modules and $\varphi\colon \mathbb{V} \to \mathbb{W}$ be a morphism consisting of isomorphisms modulo $\mathcal{D}$. Then all $\varphi_{*,\bullet}\colon H_\bullet(\mathbb{V}) \to H_\bullet(\mathbb{W})$ are also isomorphisms modulo $\mathcal{D}$.*

Proof. Consider the following commuting exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker\partial_n \arrow[r, hook] \arrow[d, "\varphi_n"'] & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & \operatorname{ran}\partial_n \arrow[r] \arrow[d, "\varphi_{n-1}"'] & 0 \\
0 \arrow[r] & \ker\partial'_n \arrow[r, hook] & W_n \arrow[r, "\partial'_n"] & \operatorname{ran}\partial'_n \arrow[r] & 0
\end{tikzcd}$$

Since φ_n is an isomorphism modulo $\mathcal{D}$ and $\ker\varphi_{n-1} \cap \operatorname{ran}\partial_n$ is in $\mathcal{D}$, Snake Lemma implies that other two column morphisms are also isomorphisms modulo $\mathcal{D}$. Now, applying Snake

Lemma again to the following commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{ran}\partial_{n+1} \arrow[r, hook] \arrow[d, "\varphi_n"] & \ker\partial_n \arrow[r] \arrow[d, "\varphi_n"] & H_n(\mathbb{V}) \arrow[r] \arrow[d, "\varphi_{*,n}"] & 0 \\ 0 \arrow[r] & \operatorname{ran}\partial_{n+1} \arrow[r, hook] & \ker\partial_n \arrow[r] & H_n(\mathbb{W}) \arrow[r] & 0 \end{tikzcd}$$

one sees that $\varphi_{*,n}$ is an isomorphism modulo $\mathcal{D}$. □

5.2. **Dimension function (after Lück).** Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and recall that $\mathrm{Proj}(\mathcal{M})$ is a lattice such that $\tau(p)+\tau(q)=\tau(p\vee q)+\tau(p\wedge q)$ for every $p,q\in\mathrm{Proj}(\mathcal{M})$. Throughout this section, a module means a left $\mathcal{M}$-module. Note that

$$\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})=\mathbb{M}_{m,n}(\mathcal{M}) \text{ by the right multiplication.}$$

Definition 5.11. A module V is *finitely generated and projective* (abbreviated as f.g.p.) if $V\cong\mathcal{M}^{\oplus m}P$ for some $m\in\mathbb{N}$ and some idempotent $P\in\mathbb{M}_m(\mathcal{M})$.

Remark 5.12. In the original definition, a module V is *projective* if every surjection onto it splits. We note that a concrete realization $\mathcal{M}^{\oplus m}P$ of V is *not* among the structures of V. We can take P to be self-adjoint, because if we set $P_0=l(P)$, then $P=SP_0S^{-1}$ for $S=I+P_0-P$. For the following, we generally assume that P is self-adjoint.

A ring R is said to be "semi-hereditary" if every f.g. R-submodule of a free R-module is projective. Every von Neumann algebra has this property.

Lemma 5.13. *(1) Every weakly closed submodule V of $\mathcal{M}^{\oplus m}$ is of the form $\mathcal{M}^{\oplus m}P$.*
(2) For every $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})$, both $\ker\varphi$ and $\operatorname{ran}\varphi$ are f.g.p.
(3) Every f.g. submodule V of $\mathcal{M}^{\oplus m}$ is projective.

Proof. Ad(1): One observes that $V=\mathcal{M}^{\oplus m}P$ for the orthogonal projection P in $\mathbb{M}_m(\mathcal{M})$ from $L^2\mathcal{M}^{\oplus m}$ onto the L^2-norm closure of V.
Ad(2): $\ker\varphi=\mathcal{M}^{\oplus m}P$ by (1) and $\operatorname{ran}\varphi\cong\mathcal{M}^{\oplus m}P^{\perp}$ by Isomorphism Theorem.
Ad(3): If V is f.g., then $V=\operatorname{ran}\varphi$ for some $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus n},\mathcal{M}^{\oplus m})$. □

Definition 5.14. For a f.g.p. module $V\cong\mathcal{M}^{\oplus m}P$, define $\dim_{\mathcal{M}}V=(\mathrm{Tr}\otimes\tau)(P)$.

Remark 5.15. The $\mathcal{M}$-dimension $\dim_{\mathcal{M}}V$ is well-defined: If $\mathcal{M}^{\oplus m}P\cong\mathcal{M}^{\oplus n}Q$, then $(\mathrm{Tr}\otimes\tau)(P)=(\mathrm{Tr}\otimes\tau)(Q)$. In particular, if $W\cong V$ (resp. $W\subset V$) are f.g.p. modules, then $\dim_{\mathcal{M}}W=\dim_{\mathcal{M}}V$ (resp. $\dim_{\mathcal{M}}W\le\dim_{\mathcal{M}}V$).

Definition 5.16. For every module V, we define the $\mathcal{M}$-dimension of V by

$$\dim_{\mathcal{M}}V=\sup\{\dim_{\mathcal{M}}W : W\subset V \text{ f.g.p. submodule}\}\in[0,\infty].$$

Note that the definitions are consistent for f.g.p. modules. The dimension function is *continuous* in the following sense: if $V=\bigcup V_i$ is a directed union of modules, then one has $\dim_{\mathcal{M}}V=\lim\dim_{\mathcal{M}}V_i$.

For $V\subset\mathcal{M}^{\oplus m}$, we denote by $\overline{V}$ the weak closure of V. Although there is a way defining $\overline{V}$ purely algebraically for arbitrary module V, we do not elaborate it.

Proposition 5.17. *Let $V\subset\mathcal{M}^{\oplus m}$ be a submodule with $\overline{V}=\mathcal{M}^{\oplus m}P$. Then, there exists a net of projections $P_i\in\mathbb{M}_m(\mathcal{M})$ such that $\mathcal{M}^{\oplus m}P_i\subset V$ and $P_i\to P$. In particular, one has $\dim_{\mathcal{M}}V=\dim_{\mathcal{M}}\overline{V}$.*

Proof. Let $V \subset \mathcal{M}^{\oplus m}$ be given. Let $i = (W, \varepsilon)$ be a pair of f.g. submodule $W \subset V$ and $\varepsilon > 0$. We choose $n \in \mathbb{N}$ and $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $W = \mathcal{M}^{\oplus n}T$, and $\delta > 0$ such that $P_i = \chi_{[\delta,1]}(T^*T) \in \mathbb{M}_m(\mathcal{M})$ satisfies $\tau(r(T) - P_i) < \varepsilon$. Since $P_i = ST$ for $S = \chi_{[\delta,1]}(T^*T)(T^*T)^{-1}T^* \in \mathbb{M}_{m,n}(\mathcal{M})$, we have $\mathcal{M}^{\oplus m}P_i \subset \mathcal{M}^{\oplus n}T \subset V$. It is not hard to see $P_i \nearrow P$. This implies that $\dim_{\mathcal{M}} V \geq \sup \dim_{\mathcal{M}} \mathcal{M}^{\oplus m}P_i = \dim_{\mathcal{M}} \overline{V}$. The converse inequality is trivial. □

Theorem 5.18 (Lück). *For every short exact sequence* $0 \to V_2 \xrightarrow{\iota} V_1 \xrightarrow{\pi} V_0 \to 0$, *one has* $\dim_{\mathcal{M}} V_1 = \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$.

Proof. Let $W \subset V_0$ be any f.g.p. submodule. Then, one has $\pi^{-1}(W) \cong W \oplus \iota(V_2)$ by the projectivity of W. Hence,

$$\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} \pi^{-1}(W) \geq \dim_{\mathcal{M}} W + \dim_{\mathcal{M}} \iota(V_2).$$

Taking the supremum over all $W \subset V_0$, one gets $\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$. In particular, we have proved that $\dim_{\mathcal{M}}$ decreases under a surjection. To prove the converse inequality, let $W \subset V_1$ be any f.g.p. submodule. We realize W as $\mathcal{M}^{\oplus m}P$. Then, one has $\overline{\iota(V_2) \cap W} = \mathcal{M}^{\oplus m}Q$ for some projection $Q \in \mathbb{M}_m(\mathcal{M})$ with $Q \leq P$. This implies that $W/\overline{\iota(V_2) \cap W} \cong \mathcal{M}^{\oplus m}(P - Q)$. It follows by Proposition 5.17 that

$$\begin{aligned}\dim_{\mathcal{M}} W &= \dim_{\mathcal{M}} W/\overline{\iota(V_2) \cap W} + \dim_{\mathcal{M}} \overline{\iota(V_2) \cap W} \\ &\leq \dim_{\mathcal{M}} W/(\iota(V_2) \cap W) + \dim_{\mathcal{M}} \iota(V_2) \cap W \\ &\leq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} \iota(V_2),\end{aligned}$$

where we have applied the first part to $W/(\iota(V_2) \cap W) \twoheadrightarrow W/\overline{\iota(V_2) \cap W}$. □

We call V a *torsion module* if $\dim_{\mathcal{M}} V = 0$. Torsion modules form a Serre subcategory and every module V has the unique largest torsion submodule $V_T \subset V$.

Corollary 5.19. *For every f.g. module* V, *one has* $V \cong V_P \oplus V_T$, *where* V_P *is f.g.p. with* $\dim_{\mathcal{M}} V_P = \dim_{\mathcal{M}} V$.

Proof. We prove that the f.g. module $V_P = V/V_T$ is projective (and hence there is a splitting $V_P \hookrightarrow V$). Take a surjection $\varphi\colon \mathcal{M}^{\oplus m} \twoheadrightarrow V_P$. Since $\overline{\ker\varphi}/\ker\varphi$ is a torsion submodule of $\mathcal{M}^{\oplus m}/\ker\varphi \cong V_P$, it is zero. It follows that $\ker\varphi$ is closed and $V_P \cong \mathcal{M}^{\oplus m}/\ker\varphi$ is projective. □

Although we do not use it explicitly, this corollary, in combination with continuity, is useful to reduce the proof of dimensional equations to those for f.g.p. modules.

Definition 5.20. A morphism $\varphi\colon V \to W$ is a $\dim_{\mathcal{M}}$*-isomorphism* if it is an isomorphism modulo torsion modules, i.e., $\dim_{\mathcal{M}} \ker\varphi = 0 = \dim_{\mathcal{M}} \operatorname{coker}\varphi$.

Lemma 5.21. *The morphism* $\mathcal{M} \hookrightarrow L^2\mathcal{M}$ *is a* $\dim_{\mathcal{M}}$*-isomorphism.*

Proof. Let $\xi \in L^2\mathcal{M}$ be given. We view it as a closed square-integrable operator affiliated with $\mathcal{M}$. Then, for $p_n = \chi_{[0,n]}(\xi\xi^*) \in \mathcal{M}$, one has $p_n \to 1$ and $p_n\xi \in \mathcal{M}$. We note that $p_n\xi \in \mathcal{M}$ means that $p_n\xi = 0$ in $L^2\mathcal{M}/\mathcal{M}$. □

Remark 5.22. From this lemma, one observes that $\dim_{\mathcal{M}}$ agrees with the von Neumann dimension function for normal Hilbert $\mathcal{M}$-modules.

5.3. Definition of the ℓ_2-Betti numbers (after Lück).

Definition 5.23. For a discrete group Γ, we define the n-th ℓ_2-Betti number of Γ by

$$\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \mathrm{Tor}_n^{\mathbb{C}\Gamma}(\mathcal{L}\Gamma, \mathbb{C}),$$

where $\mathbb{C}$ is the trivial $\mathbb{C}\Gamma$-module: $f \cdot z = \sum_{s\in\Gamma} f(s)z$.

Exercise. Prove that $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \mathrm{Tor}_n^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C})$. (Hint: You have to show that the functor $\ell_2\Gamma \otimes_{\mathcal{L}\Gamma} \cdot$ is exact and $\dim_{\mathcal{L}\Gamma}$-preserving.)

Example. For $d = 1, 2, \ldots$, one has

$$\beta_n^{(2)}(\mathbb{F}_d) = \begin{cases} d-1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $g_1, \ldots, g_d$ be the canonical generators of $\mathbb{F}_d$. We consider the complex

$$\mathbb{V}: \qquad 0 \longrightarrow (\mathbb{C}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathbb{C}\mathbb{F}_d \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0,$$

where $\partial_0(\xi) = \sum_{s\in\mathbb{F}_d} \xi(s)$ and $\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i$. (We define $(\xi \cdot s)(t) = \xi(ts)$.) We will show that the complex $\mathbb{V}$ is exact. We check $\ker \partial_1 = 0$. Let $\chi_j \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words starting at g_j. It is not hard to see that $\chi_j \cdot g_i^{-1} = \chi_j + \delta_{i,j}\delta_e$ for every i, j. If $(\xi_i)_{i=1}^d \in \ker \partial_1$, then for every $s \in \Gamma$ and j, one has

$$0 = \langle \sum_{i=1}^d \xi_i \cdot g_i - \xi_i,\, s \cdot \chi_j \rangle = \sum_{i=1}^d \langle \xi_i,\, s \cdot (\chi_j \cdot g_i^{-1} - \chi_j) \rangle = \xi_j(s)$$

and $(\xi_i)_{i=1}^d = 0$. We next check $\operatorname{ran} \partial_1 = \ker \partial_0$. It is easy to see $\partial_0 \circ \partial_1 = 0$. Let $\chi_i^\vee \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words ending at g_i^{-1}. We observe that $\chi_i - s \cdot \chi_i^\vee$ is finitely supported for every $s \in \mathbb{F}_d$. (Indeed, it suffices to check this for $g_1, \ldots, g_d$.) Moreover, since $\chi_i^\vee \cdot g_i - \chi_i^\vee$ is the characteristic function of the reduced words ending at other than $g_i^{\pm 1}$, one has $\sum_{i=1}^d \chi_i^\vee \cdot g_i - \chi_i^\vee = (d-1)\mathbf{1} + \delta_e$. Now, suppose $\xi \in \ker \partial_0$. Then, since

$$\xi = -(\sum_{s\neq e} \xi(s)\delta_e) + \sum_{s\neq e} \xi(s)\delta_s = \sum_{s\neq e} \xi(s)(\delta_e - \delta_s),$$

$\xi_i = \xi * \chi_i^\vee \in \mathbb{C}\Gamma$ by the above observation, and since $\xi * \mathbf{1} = 0$, one has

$$\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i = \sum_{i=1}^d \xi * (\chi_i^\vee \cdot g_i - \chi_i^\vee) = \xi.$$

We have proved that $\mathbb{V}$ is a projective resolution of $\mathbb{C}$. Since

$$\mathcal{L}\mathbb{F}_d \otimes_{\mathbb{C}\mathbb{F}_d} \mathbb{V}_{\geq 0}: \qquad 0 \longrightarrow (\mathcal{L}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathcal{L}\mathbb{F}_d \longrightarrow 0,$$

one has

$$\mathrm{Tor}_n^{\mathbb{C}\mathbb{F}_d}(\mathcal{L}\mathbb{F}_d, \mathbb{C}) = \begin{cases} \mathcal{L}\mathbb{F}_d / \operatorname{ran} \partial_1 & \text{if } n = 0 \\ \ker \partial_1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Since $\lambda(s) - \lambda(t) \in \operatorname{ran} \partial_1$ and $\lambda(t) \to 0$ weakly as $t \to \infty$, one has $\lambda(s) \in \overline{\operatorname{ran} \partial_1}$ for every $s \in \mathbb{F}_d$ and hence $\overline{\operatorname{ran} \partial_1} = \mathcal{L}\mathbb{F}_d$. It follows that $\beta_0^{(2)}(\mathbb{F}_d) = 0$ and $\beta_1^{(2)}(\mathbb{F}_d) = \dim_{\mathcal{L}\mathbb{F}_d} \ker \partial_1 = d - \dim_{\mathcal{L}\mathbb{F}_d} \operatorname{ran} \partial_1 = d - 1$. $\square$

Below, we sketch an argument showing that the above definition of ℓ_2-Betti numbers is consistent with another(?). We denote by $\mathcal{F}(\Gamma, X)$ the set of functions from a set Γ into X. Now Γ be a discrete group and consider ℓ_Γ as a right Γ-module. The is a natural complex

$$0 \longrightarrow \ell_2\Gamma \xrightarrow{\partial_0} \mathcal{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial_1} \mathcal{F}(\Gamma^2, \ell_2\Gamma) \longrightarrow \cdots,$$

where $(\partial_0 f)(s) = f - f \cdot s$ and $(\partial_1 b)(s,t) = b(t) - b(st) + b(s) \cdot t$, etc. We then define the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ by $H_n(\Gamma, \ell_2\Gamma) = \ker \partial_n / \operatorname{ran} \partial_{n+1}$. Since ∂_n commutes with the $\mathcal{L}\Gamma$-action on $\ell_2\Gamma$, the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ is naturally an $\mathcal{L}\Gamma$-module. We define $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} H_n(\Gamma, \ell_2\Gamma)$. Let us calculate $\beta_n^{(2)}(\Gamma)$ for $n = 0, 1$. Since $H_0 \subset \ell_2\Gamma$ is the subspace of constant functions, one has $\beta_0^{(2)}(\Gamma) = |\Gamma|^{-1}$. We note that $D(\Gamma) = \ker \partial_1$ is the space of derivations and $D_0(\Gamma) = \operatorname{ran} \partial_0$ is the space of inner derivations. To see what $\beta_1^{(2)}(\Gamma)$ is, we assume that Γ is generated by a finite subset $\{s_1, \ldots, s_d\}$. Then, there is an $\mathcal{L}\Gamma$-module map

$$D(\Gamma) \ni b \longmapsto (b(s_i))_{i=1}^d \in \bigoplus_{i=1}^d \ell_2\Gamma,$$

which is an isomorphism onto a closed subspace. We note that $D_0(\Gamma)$ is closed in $\bigoplus_{i=1}^d \ell_2\Gamma$ iff Γ is finite or non-amenable, and that $\dim_{\mathcal{L}\Gamma} \overline{D_0(\Gamma)} = \dim_{\mathcal{L}\Gamma}(\ker \partial_0)^\perp = 1 - |\Gamma|^{-1}$. Hence, one has $\dim_{\mathcal{L}\Gamma} D(\Gamma) = \beta_1^{(2)}(\Gamma) + \dim_{\mathcal{L}\Gamma} D_0(\Gamma) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$. We view ∂_0 as a map from $\ell_2\Gamma$ into $\bigoplus_{i=1}^d \ell_2\Gamma$ and consider

$$\partial_0^*\colon \bigoplus_{i=1}^d \ell_2\Gamma \ni (\xi_i) \longmapsto \sum_i \xi_i - \xi_i \cdot s_i^{-1} \in \ell_2\Gamma.$$

Lemma 5.24. *One has* $(\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp = D(\Gamma)$.

Proof. We note that the scalar product $\langle \cdot, \cdot \rangle$ is defined consistently on $\mathbb{C}\Gamma \times \mathcal{F}(\Gamma, \mathbb{C})$ and on $\ell_2\Gamma \times \ell_2\Gamma$. Moreover, $\mathcal{F}(\Gamma, \mathbb{C})$ is the algebraic dual of $\mathbb{C}\Gamma$ w.r.t. this scalar product. Suppose that $b \in D(\Gamma)$. It is not hard to show that $b = f - f \cdot s$ for some $f \in \mathcal{F}(\Gamma, \mathbb{C})$. It follows that for every $\xi \in \ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma$, one has

$$\langle \xi, b \rangle = \sum \langle \xi_i, b(s_i) \rangle = \sum_i \langle \xi_i - \xi_i \cdot s_i^{-1}, f \rangle = 0.$$

Conversely, if $b \in \ell_2\Gamma$ is such that $b \perp (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)$, then the linear functional $\langle \cdot, b \rangle$ on $\bigoplus \mathbb{C}\Gamma$ factors through ∂_0^* and there is $f \in \mathcal{F}(\Gamma, \mathbb{C})$ such that $\langle \xi, b \rangle = \langle \partial_0^*(\xi), f \rangle$ for every $\xi \in \bigoplus \mathbb{C}\Gamma$. It follows that $b(s) = f - f \cdot s$ and $b \in D(\Gamma)$. $\square$

Since $(\ker \partial_0^*)^\perp = \overline{\operatorname{ran} \partial_0} = \overline{D_0(\Gamma)}$, one has

$$\begin{aligned} D(\Gamma)/\overline{D_0(\Gamma)} &\cong (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \ominus (\ker \partial_0^*)^\perp \\ &= \ker \partial_0^* \cap (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \quad \cong \operatorname{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}). \end{aligned}$$

The last isomorphism follows from the following observation:

$$\mathbf{V}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^d \mathbb{C}\Gamma \xrightarrow{\partial_0^*} \mathbb{C}\Gamma \longrightarrow \mathbb{C}$$

is a free resolution of the trivial left $\mathbb{C}\Gamma$-module $\mathbb{C}$ and

$$\ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \mathbb{V}_{\geq 0}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^{d} \ell_2\Gamma \xrightarrow{\partial_0^*} \ell_2\Gamma \longrightarrow 0$$

with $\operatorname{ran}\partial_1^* = \ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma}) \subset \overline{\mathbb{C}\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})} = \overline{\ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})}$.

5.4. Rank metric (after Thom).

Definition 5.25. Let V be a left $\mathcal{M}$-module. For $\xi \in V$, we define its *rank norm* by

$$[\xi] = \inf\{\tau(p) : p \in \operatorname{Proj}(A),\ p\xi = \xi\} \in [0,1].$$

We record several basic properties of the rank norm.

Lemma 5.26. *For a left $\mathcal{M}$-module V, the following are true.*
(1) Triangle inequality: $[\xi + \eta] \leq [\xi] + [\eta]$ for every $\xi, \eta \in V$.
(2) $[x\xi] \leq \min\{[x], [\xi]\}$ for every $x \in \mathcal{M}$ and $\xi \in V$.
(3) $V_T = \{\xi \in V : [\xi] = 0\}$.
(4) A submodule $W \subset V$ is dense in rank norm if and only if $\dim_{\mathcal{M}} V/W = 0$.
(5) Every $\varphi \in \operatorname{Mor}(V, W)$ is a rank contraction: $[\varphi(\xi)] \leq [\xi]$.
(6) For every $\varphi \in \operatorname{Mor}(V, W)$, $\eta \in \operatorname{ran}\varphi$ and $\varepsilon > 0$, there exists $\xi \in \varphi^{-1}(\eta)$ such that $[\eta] \leq [\xi] + \varepsilon$.

Proof. The triangle inequality follows from the fact that $\tau(p \vee q) \leq \tau(p) + \tau(q)$. The second assertion follows from the fact that $p\xi = \xi$ implies $xp\xi = \xi$ and $[x\xi] \leq \tau(l(xp)) \leq \tau(p)$. For the third assertion, we observe that $[\xi] = 0$ iff $\mathcal{M}\xi$ is a torsion submodule. Indeed, the "if" part is rather easy and the "only if" part follows by considering the morphism $\varphi\colon M \ni x \mapsto x\xi \in V$. Since $\ker\varphi$ is a left ideal with $\dim_{\mathcal{M}} \ker\varphi = 1$, i.e., $\overline{L} = \mathcal{M}$, Proposition 5.17 implies that there is a net $p_i \in L$ of projections such that $p_i \to 1$. This means $[\xi] = 0$. The rest are trivial. □

We recall that the *completion* of a metric space (X, d) is the metric space of all equivalence classes of Cauchy sequences in X. Here, two Cauchy sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are equivalent if $d((x_n), (y_n)) := \lim_n d(x_n, y_n) = 0$.

Definition 5.27. The *rank completion* of a left $\mathcal{M}$-module V is the completion $C(V)$ of V w.r.t. the *rank metric* d, where $d(\xi, \eta) = [\xi - \eta]$ for $\xi, \eta \in V$. We observe that

$$C(V) = \{\text{ Cauchy sequences in V }\}/\{\text{ Null sequences }\}$$

and that $C(V)$ is naturally a left $\mathcal{M}$-module (thanks to Lemma 5.26).

The rank metric is actually a pseudo-metric. More precisely, it is a metric on V/V_T. The constant "embedding" $c\colon V \to C(V)$ is a $\dim_{\mathcal{M}}$-isomorphism and it induces a canonical inclusion $V/V_T \hookrightarrow C(V)$. Moreover, $C(V)$ is the unique torsion-free complete $\mathcal{M}$-module containing V/V_T as a dense submodule. Indeed, one has:

Lemma 5.28. *Let V, W be $\mathcal{M}$-modules with W torsion-free and complete. Then, every $\varphi \in \operatorname{Mor}(V, W)$ extends to $\tilde{\varphi} \in \operatorname{Mor}(C(V), W)$, i.e., $\tilde{\varphi} \circ c = \varphi$.*

Proposition 5.29. *The rank completion c is an exact functor.*

Proof. Let a short exact sequence $0 \longrightarrow V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \longrightarrow 0$ be given.

Exactness at $C(V_0)$. Let $\xi \in C(V_0)$ and choose a representing Cauchy sequence $(\xi_n)_n$ in V_0 such that $d(\xi_n, \xi_{n+1}) < 2^{-(n+1)}$. We will construct $\eta_1, \eta_2, \ldots$ such that $\partial_1(\eta_n) = \xi_n$ and $d(\eta_n, \eta_{n+1}) < 2^{-n}$. Suppose $\eta_1, \ldots, \eta_n$ have been chosen. Lift $\xi_{n+1} - \xi_n \in V_0$ to $\zeta_{n+1} \in V_1$ with $[\zeta_{n+1}] \leq [\xi_{n+1} - \xi_n] + 2^{-(n+1)}$. Set $\eta_{n+1} = \eta_n + \zeta_{n+1}$ and we are done. Now the sequence $(\eta_n)_n$ is Cauchy in V_1 and hence converges to an element η in $C(V_1)$ such that $\partial_1(\eta) = \xi$.

Exactness at $C(V_1)$. It is clear that $C(\partial_1) \circ C(\partial_2) = 0$ by continuity. Let $\xi \in \ker C(\partial_1)$ be given and choose $(\xi_n)_n$ in V_1 such that $\xi_n \to \xi$. Since $\partial_1(\xi_n) \to C(\partial_1)(\xi) = 0$, the sequence $(\partial_1(\xi_n))_n$ is null. Hence, one can lift $(\partial_1(\xi_n))_n$ to a null sequence $(\eta_n)_n$ in V_1. It follows that $(\xi_n - \eta_n)_n$ is a Cauchy sequence in $\ker \partial_1 = \operatorname{ran} \partial_2$. Therefore,

$$\xi = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} (\xi_n - \eta_n) \in \overline{\operatorname{ran} \partial_2} = \operatorname{ran} C(\partial_2),$$

where we used the result of the previous paragraph for the last equality.

Exactness at $C(V_2)$. Since ∂_2 is an isometry, $C(\partial_2)$ is an isometry as well. Since $C(V_2)$ does not have a non-zero torsion element, $C(\partial_2)$ is injective. □

5.5. Gaboriau's theorem (after Sauer and Thom).

Proposition 5.30. *Let $\mathcal{M} \subset \mathcal{N}$ be finite von Neumann algebras with $\tau_{\mathcal{M}} = \tau_{\mathcal{N}}|_{\mathcal{M}}$. Then, $\mathcal{N}$ is a flat $\mathcal{M}$-module and $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for any $\mathcal{M}$-module V.*

Proof. We use Lemma 5.9 to prove flatness. Let $V \subset \mathcal{M}^{\oplus m}$ be a f.g. submodule. It follows that there is $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $V = \mathcal{M}^{\oplus n}T$. Let P be the left support of T and observe that $V = \mathcal{M}^{\oplus n}T \ni \xi T \mapsto \xi P \in \mathcal{M}^{\oplus n}P$ is an isomorphism. Since $\mathcal{M}^{\oplus n}P$ is a direct summand of $\mathcal{M}^{\oplus n}$, one has the following kosher identifications

$$\mathcal{N} \otimes_{\mathcal{M}} V \cong \mathcal{N} \otimes_{\mathcal{M}} (\mathcal{M}^{\oplus n}P) \cong \mathcal{N}^{\oplus n}P \cong \mathcal{N}^{\oplus n}T \subset \mathcal{N}^{\oplus m} \cong \mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}^{\oplus m}.$$

It follows from Lemma 5.9 that $\mathcal{N}$ is flat.

Since the dimension function is continuous w.r.t. inductive limits, it suffices to check the identity $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for a f.g. V. Since $\mathcal{M}^{\oplus m}P \hookrightarrow V$ implies $\mathcal{N}^{\oplus m}P \hookrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$, one has $\dim_{\mathcal{M}} V \leq \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$. To prove the converse inequality, take a surjection $\pi \colon \mathcal{M}^{\oplus n} \twoheadrightarrow V$. Then, $\mathrm{id} \otimes \pi \colon \mathcal{N}^{\oplus n} \twoheadrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$ is also a surjection such that $\ker(\mathrm{id} \otimes \pi) = \mathcal{N} \otimes_{\mathcal{M}} \ker \pi$ by flatness. It follows that

$$\dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V = n - \dim_{\mathcal{N}} \ker(\mathrm{id} \otimes \pi) \leq n - \dim_{\mathcal{M}} \ker \pi = \dim_{\mathcal{M}} V$$

by the previous inequality. □

Let $\Gamma \curvearrowright (X, \mu)$ be an essentially-free probability-measure-preserving action. Let $\mathcal{A} = L^\infty(X, \mu)$, $\mathcal{M} = \mathcal{L}\Gamma$ and $\mathcal{N} = \mathcal{A} \rtimes \Gamma$. Let $R_0 \subset \mathcal{N}$ (resp. $R \subset \mathcal{N}$) be the $\mathbb{C}$-algebra generated by $\mathcal{A}$ and Γ (resp. by $\mathcal{A}$ and the full group $[\Gamma]$). Then,

$$\mathcal{A} \subset R_0 \subset R \subset \mathcal{N}$$

and $\mathcal{A}$ is a left R-module: $a\varphi \cdot f = a\varphi_*(f)$ for $a, f \in \mathcal{A}$ and $\varphi \in [\Gamma]$. Now, Gaboriau's theorem that $\beta_\bullet^{(2)}(\Gamma)$ is an invariant of $[\Gamma]$ follows from the following equalities:

$$\begin{aligned}
\beta_\bullet^{(2)}(\Gamma) &= \dim_{\mathcal{M}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{M}, \mathbb{C}) & \\
&= \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{M}, \mathbb{C}) & \text{by Proposition 5.30} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}, \mathbb{C}) & \text{since } \mathcal{N} \text{ is flat over } \mathcal{M} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{R_0}(\mathcal{N}, \mathcal{A}) & (\spadesuit) \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{R}(\mathcal{N}, \mathcal{A}) & (\heartsuit)
\end{aligned}$$

The proof of ($\spadesuit$) is rather routine: Since $\mathcal{N}$ is also a right R_0-module and R_0 is a free left $\mathbb{C}\Gamma$-module (Consider the conditional expectation onto $\mathcal{A}$), one has

$$\mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{N}, V) = \mathrm{Tor}_\bullet^{R_0}(\mathcal{N}, R_0 \otimes_{\mathbb{C}\Gamma} V)$$

for any $\mathbb{C}\Gamma$-module V. Indeed, if $\mathbb{V}$ is a projective resolution of V, then $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}$ is a projective resolution of $R_0 \otimes_{\mathbb{C}\Gamma} V$ with $\mathcal{N} \otimes_{R_0} (R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}) \cong \mathcal{N} \otimes_{\mathbb{C}\Gamma} \mathbb{V}$. We then observe that $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \cong \mathcal{A}$ as an R_0-module. The proof of ($\heartsuit$) is more involved, but reduces to the fact that $R_0 \subset R$ is dense in an appropriate sense.

We write $[\xi]_{\mathcal{A}}$ (resp. $[\xi]_{\mathcal{N}}$) for the rank norm w.r.t. $\mathcal{A}$ (resp. $\mathcal{N}$) and note that $[\xi]_{\mathcal{N}} \le [\xi]_{\mathcal{A}}$. In particular, one has $[x]_{\mathcal{A}} = \inf\{\, \tau(p) : p \in \mathrm{Proj}(\mathcal{A}),\ px = x\}$ for $x \in \mathcal{N}$. For $x \in \mathcal{N}$, we define

$$|[x]|_{\mathcal{A}} = \sup\{\, [xp]_{\mathcal{A}}/[p]_{\mathcal{A}} : p \in \mathrm{Proj}(\mathcal{A})\} \in [0, \infty].$$

We record several basic properties of this norm.

Lemma 5.31. *(1)* $|[\alpha x]|_{\mathcal{A}} = |[x]|_{\mathcal{A}}$ *for every* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $x \in \mathcal{N}$.

(2) $|[v]|_{\mathcal{A}} = 1$ *for every non-zero pseudo-normalizer* v *of* $\mathcal{A}$ *in* $\mathcal{N}$.

(3) $|[x+y]|_{\mathcal{A}} \le |[x]|_{\mathcal{A}} + |[y]|_{\mathcal{A}}$ *and* $|[xy]|_{\mathcal{A}} \le |[x]|_{\mathcal{A}} |[y]|_{\mathcal{A}}$ *for every* $x, y \in \mathcal{N}$.

(4) $|[x]|_{\mathcal{A}} < \infty$ *for every* $x \in R$.

(5) For every $x \in R$, *there is a sequence* $(x_n)_n$ *in* R_0 *such that* $[x_n - x]_{\mathcal{A}} \to 0$ *and* $\sup |[x_n]|_{\mathcal{A}} < \infty$.

(6) If V *is an* R_0*-module, then* $[x\xi]_{\mathcal{A}} \le |[x]|_{\mathcal{A}} [\xi]_{\mathcal{A}}$ *for every* $x \in R_0$ *and* $\xi \in V$. *The same thing holds for* R.

Lemma 5.32. *Let* V *be a left* R_0*-module. Then, the rank completion* $C(V)$ *w.r.t.* $\mathcal{A}$ *is naturally a left* R*-module. Moreover,* C *is a natural functor from the category of* R_0*-modules into the category of complete* R*-modules.*

Proof. By the previous lemma, one knows that $C(V)$ is naturally an R_0-module. Let $x \in R$ and $\xi \in C(V)$ be given. Choose a sequence $(x_n)_n$ in R_0 such that $[x - x_n]_{\mathcal{A}} \to 0$. Then, $(x_n\xi)_n$ is a Cauchy sequence in $C(V)$ and has a limit $x\xi$ in $C(V)$. We note that the limit is independent of the choice of $(x_n)_n$. Moreover, if $|[x_n]|_{\mathcal{A}}$ is bounded and $[y_m - y]_{\mathcal{A}} \to 0$, then $[x_n y_m - xy]_{\mathcal{A}} \to 0$. This shows $(xy)\xi = x(y\xi)$. □

Lemma 5.33. *Let* V *be a left* R_0*-module. Then the constant embedding*

$$\mathrm{id} \otimes c \colon \mathcal{N} \otimes_{R_0} V \to \mathcal{N} \otimes_{R_0} C(V)$$

is a $\dim_{\mathcal{N}}$*-isomorphism. The same thing holds for* R.

Proof. Suppose that $\sum_{i=1}^n x_i \otimes \xi_i \in \ker(\mathrm{id} \otimes c)$ and $\varepsilon > 0$ be given. Then, one has

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_j b_j r_j \otimes \eta_j - b_j \otimes r_j \eta_j \quad \text{in } \mathcal{N} \otimes_{\mathbf{C}} C(V).$$

Choose $p_j \in \mathrm{Proj}(\mathcal{A})$ such that $p_j^\perp \eta_j \in V$ and $n \sum (1 + \|[r_j]\|_{\mathcal{A}}) \tau(p_j) < \varepsilon$. It follows that there is $p \in \mathrm{Proj}(\mathcal{A})$ such that $pp_j = p_j$, $pr_j p_j = r_j p_j$ and $\tau(p) < \varepsilon/n$. Since $\sum_j b_j r_j \otimes p_j^\perp \eta_j - b_j \otimes r_j p_j^\perp \eta_j$ is zero in $\mathcal{N} \otimes_{R_0} V$, subtracting it from the both sides of the above equation, we may assume that

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_j b_j r_j \otimes p_j \eta_j - b_j \otimes r_j p_j \eta_j \quad \text{in } \mathcal{N} \otimes_{\mathbf{C}} C(V).$$

It follows that $\sum x_i \otimes \xi_i = \sum x_i \otimes p\xi_i$ in $\mathcal{N} \otimes_{\mathbf{C}} C(V)$, and *a fortiori* in $\mathcal{N} \otimes_{\mathbf{C}} V$ since $\mathcal{N} \otimes_{\mathbf{C}} V \subset \mathcal{N} \otimes_{\mathbf{C}} C(V)$ (recall any module over a field is free). Hence, one has

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_{i=1}^n x_i \otimes p\xi_i = \sum_{i=1}^n x_i p \otimes \xi_i \quad \text{in } \mathcal{N} \otimes_{R_0} V.$$

This implies that $[\sum_{i=1}^n x_i \otimes \xi_i]_{\mathcal{N}} \le \sum [x_i p]_{\mathcal{N}} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, one sees that $\ker(\mathrm{id} \otimes c)$ is a torsion submodule. That $\mathrm{ran}(\mathrm{id} \otimes c)$ is dense in $\mathcal{N} \otimes_{R_0} C(V)$ follows from the fact that $[x \otimes \xi]_{\mathcal{N}} \le [\xi]_{\mathcal{A}}$ for every $x \in \mathcal{N}$ and $\xi \in C(V)$. □

We omit the proof of the next lemma, which is similar to that of the previous one.

Lemma 5.34. *Let V be a left R-module, then the surjection*

$$N \otimes_{R_0} V \twoheadrightarrow N \otimes_R V$$

is a $\dim_{\mathcal{N}}$-isomorphism.

We are now in position to complete the proof of Gaboriau's theorem.

Proof of ($\heartsuit$). Let $\mathbf{V}$ (resp. $\mathbf{W}$) be a projective resolution of $\mathcal{A}$ as an R_0-module (resp. as an R-module). Then, by Theorem 5.3 (and Proposition 5.29), the identity morphism $\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ (resp. the constant embedding $c \colon \mathcal{A} \to C(\mathcal{A})$) extends to a morphism $\varphi \colon \mathbf{V} \to \mathbf{W}$ (resp. a morphism $\psi \colon \mathbf{W} \to C(\mathbf{V})$):

$$\begin{tikzcd}
\mathbf{V}: & \cdots \arrow[r] & V_n \arrow[r] \arrow[d, "\varphi_n"] & \cdots \arrow[r] & V_0 \arrow[r, two heads] \arrow[d, "\varphi_0"] & \mathcal{A} \arrow[d, equal, "\mathrm{id}_{\mathcal{A}}"] \\
\mathbf{W}: & \cdots \arrow[r] & W_n \arrow[r] \arrow[d, "\psi_n"] & \cdots \arrow[r] & W_0 \arrow[r, two heads] \arrow[d, "\psi_0"] & \mathcal{A} \arrow[d, "c"] \\
C(\mathbf{V}): & \cdots \arrow[r] & C(V_n) \arrow[r] \arrow[d, "\tilde{\varphi}_n"] & \cdots \arrow[r] & C(V_0) \arrow[r, two heads] \arrow[d, "\tilde{\varphi}_0"] & C(\mathcal{A}) \arrow[d, equal, "\mathrm{id}_{C(\mathcal{A})}"] \\
C(\mathbf{W}): & \cdots \arrow[r] & C(W_n) \arrow[r] & \cdots \arrow[r] & C(W_0) \arrow[r, two heads] & C(\mathcal{A})
\end{tikzcd}$$

By the uniqueness part of Theorem 5.3, the compositions $\psi \circ \varphi$ and $\tilde{\varphi} \circ \psi$ are homotopic to the morphisms of the constant embeddings. Taking tensor products, one has

$$\begin{array}{lc}
\mathcal{N} \otimes_{R_0} \mathbf{V}_{\geq 0}: & \cdots \longrightarrow \mathcal{N} \otimes_{R_0} V_n \longrightarrow \cdots \\
& \big\downarrow {\scriptstyle \mathrm{id}\otimes\varphi_n} \\
\mathcal{N} \otimes_R \mathbf{W}_{\geq 0}: & \cdots \longrightarrow \mathcal{N} \otimes_R W_n \longrightarrow \cdots \\
& \big\downarrow {\scriptstyle \mathrm{id}\otimes\psi_n} \\
\mathcal{N} \otimes_R C(\mathbf{V})_{\geq 0}: & \cdots \longrightarrow \mathcal{N} \otimes_R C(V_n) \longrightarrow \cdots \\
& \big\downarrow {\scriptstyle \mathrm{id}\otimes\tilde{\varphi}_n} \\
\mathcal{N} \otimes_R C(\mathbf{W})_{\geq 0}: & \cdots \longrightarrow \mathcal{N} \otimes_R C(W_n) \longrightarrow \cdots
\end{array}$$

The morphism from $\mathcal{N} \otimes_{R_0} \mathbf{V}_{\geq 0}$ to $\mathcal{N} \otimes_R C(\mathbf{V})_{\geq 0}$ and the morphism from $\mathcal{N} \otimes_R \mathbf{W}_{\geq 0}$ to $\mathcal{N} \otimes_R C(\mathbf{W})_{\geq 0}$ are homotopic to the morphisms of constant embeddings. Since constant embeddings are $\dim_{\mathcal{N}}$-isomorphisms by Lemmas 5.33 and 5.34, the induced morphisms on the homology modules are all $\dim_{\mathcal{N}}$-isomorphisms by Lemma 5.10. It follows that $\varphi_{*,\bullet}\colon \mathrm{Tor}_\bullet^{R_0}(\mathcal{N}, \mathcal{A}) \to \mathrm{Tor}_\bullet^{R}(\mathcal{N}, \mathcal{A})$ are all $\dim_{\mathcal{N}}$-isomorphisms. □

Let $p \in \mathcal{N}$ be a projection and V be an $\mathcal{N}$-module. It is not hard to check that $p\mathcal{N} \otimes_{\mathcal{N}} V \cong pV$ and $\dim_{p\mathcal{N}p} p\mathcal{N} \otimes_{\mathcal{N}} V = \tau(p)^{-1} \dim_{\mathcal{N}} V$, where one uses the normalized trace $\tau(p)^{-1}\tau(\,\cdot\,)$ for $p\mathcal{N}p$. If $p \in \mathrm{Proj}(\mathcal{A})$ is a projection such that $\sum_i v_i p v_i^* = 1$ for some pseudo-normalizers $v_1, \ldots, v_n$, then $\mathcal{N} \otimes_R V \cong \mathcal{N}p \otimes_{pRp} pV$ for every R-module V. whose central support in $\mathcal{N}$ is 1. It follows that

$$\begin{aligned}
\dim_{\mathcal{N}} \mathrm{Tor}_\bullet^R(\mathcal{N}, \mathcal{A}) &= \tau(p) \dim_{p\mathcal{N}p} p\mathcal{N} \otimes_{\mathcal{N}} \mathrm{Tor}_\bullet^R(\mathcal{N}, \mathcal{A}) \\
&= \tau(p) \dim_{p\mathcal{N}p} \mathrm{Tor}_\bullet^{pRp}(p\mathcal{N}p, p\mathcal{A}).
\end{aligned}$$

With little more analysis, one can show the above equation for every $p \in \mathrm{Proj}(\mathcal{A})$ with full central support.

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